

LOCAL COHOMOLOGY OF BIGRADED REES ALGEBRAS, BHATTACHARYA COEFFICIENTS AND JOINT REDUCTIONS

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ABSTRACT. Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2, I and J \mathfrak{m} -primary ideals in R and (a, b) a joint reduction of (I, J) . In this paper we consider the problem whether $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r, s)}$ has finite length as an R -module, where \mathcal{R}' denotes the bigraded extended Rees algebra of I and J . We give an example to show that the answer is negative in general. For all $r, s \geq 0$, we give equivalent conditions for $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r, s)}$ to have finite length in terms of the first Bhattacharya coefficients $e_{(1, 0)}$ and $e_{(0, 1)}$ and the first Hilbert coefficient $e_1(I)$ and $e_1(J)$. Here (a, b) is a joint reduction of (I, J) satisfying superficial conditions. As a consequence we prove that $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(0, 0)}$ has finite length if and only if $e_{(1, 0)} = e_1(I)$ and $e_{(0, 1)} = e_1(J)$. We give necessary and sufficient conditions for vanishing of $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r, s)}$ in terms of Bhattacharya coefficients and Hilbert coefficients.

1. INTRODUCTION

Throughout this paper (R, \mathfrak{m}) is a Noetherian local ring with infinite residue field. Let I, J be \mathfrak{m} -primary ideals in R . Let \bar{I} denote the integral closure of an ideal I in R . For indeterminates t_1 and t_2 over R , let $\mathcal{R}' := \bigoplus_{r, s \in \mathbb{Z}} I^r J^s t_1^r t_2^s$ (resp. $\overline{\mathcal{R}'} = \bigoplus_{r, s \in \mathbb{Z}} \bar{I}^r \bar{J}^s t_1^r t_2^s$) $\in R[t_1, t_2, t_1^{-1}, t_2^{-1}]$ be the bigraded extended Rees algebra of the filtration $\{I^r J^s\}_{r, s \in \mathbb{Z}}$ (resp. $\{\bar{I}^r \bar{J}^s\}_{r, s \in \mathbb{Z}}$). In [9] the second author and J. Verma derived a formula for $\lambda_R([H^2_{(at_1, bt_2)}(\overline{\mathcal{R}'})]_{(r, s)})$, for all $r, s \geq 0$, in terms of the normal Hilbert coefficients of I and J which shows that $\lambda_R([H^2_{(at_1, bt_2)}(\overline{\mathcal{R}'})]_{(r, s)}) < \infty$ in an analytically unramified Cohen-Macaulay local ring of dimension 2 for a good joint reduction (a, b) of $\{\bar{I}^r \bar{J}^s\}_{r, s \in \mathbb{Z}}$ [9, Theorem 3.7]. Motivated by this result we ask:

Question 1.1. If R is a Cohen-Macaulay local ring of dimension 2 and (a, b) a joint reduction of (I, J) , then is $\lambda_R([H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r, s)}) < \infty$?

This question does not have a positive answer in general. We give an example to show that $\lambda_R([H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r, s)})$ is not finite (Example 4.10). For any joint reduction (a, b) of (I, J) satisfying superficial conditions we give equivalent criterion for $\lambda(H^2_{(at_1, bt_2)}(\mathcal{R}')_{(r, s)})$ to be finite (Theorem 4.3). As a consequence we prove that if $\lambda([H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(r, s)}) < \infty$ for some $r, s \geq 0$ then $\lambda([H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(p, q)}) < \infty$ for all $p \geq r$ and $q \geq s$ (Theorem 4.7).

Key words : Hilbert coefficients, Bhattacharya coefficients, joint reductions, Rees algebra, local cohomology, joint reduction number zero.

Let $\lambda(M)$ denote the length of an R -module M . Let $d = \dim R$. There exists a polynomial $P_I(x) \in \mathbb{Q}[x]$, called the *Hilbert-Samuel polynomial* of I , such that $\lambda(R/I^n) = P_I(n)$ for $n \gg 0$. This polynomial can be written as

$$P_I(x) = e_0(I) \binom{x+d-1}{d} - e_1(I) \binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(I)$$

for some integers $e_i(I)$, for $i = 0, \dots, d$, known as the *Hilbert coefficients* of I . The coefficient $e_0(I)$, which we will denote by $e(I)$, is the *multiplicity* of I . P. B. Bhattacharya showed that there exists a polynomial $P_{I,J}(x, y) \in \mathbb{Q}[x, y]$ such that $\lambda(R/I^r J^s) = P_{I,J}(r, s)$ for $r, s \gg 0$ [1, Theorem 8]. We write

$$P_{I,J}(x, y) = \sum_{i+j \leq d} (-1)^{d-(i+j)} e_{(i,j)}(I, J) \binom{x+i-1}{i} \binom{y+j-1}{j}$$

for some integers $e_{(i,j)}(I, J)$ called as the *Bhattacharya coefficients* of I and J . We set $e_{(i,j)} = e_{(i,j)}(I, J)$ if the ideals I and J are clear from the context.

We have

$$\lambda\left(\frac{R}{I^r J^s}\right) = \lambda\left(\frac{R}{I^r}\right) + \lambda\left(\frac{I^r}{I^r J^s}\right) = \lambda\left(\frac{R}{J^s}\right) + \lambda\left(\frac{J^s}{I^r J^s}\right). \quad (1.2)$$

This raises the question :

Question 1.3. Does there exist a relationship between Bhattacharya coefficients and Hilbert coefficients ?

Let $d = 2$. All the coefficients of terms degree two and the constant term of the polynomial $P_{I,J}(x, y)$ can be expressed in terms of the Hilbert coefficients of $P_I(x)$ and $P_J(x)$. In particular, $e_{(2,0)} = e(I)$ and $e_{(0,2)} = e(J)$ ([14, Theorem 2.4]). Since $P_{IJ}(x) = P_{I,J}(x, x)$, comparing the coefficient of degree two we get $e_{(1,1)} = \frac{1}{2}[e(IJ) - e(I) - e(J)]$ and $e_{(0,0)} = e_2(IJ)$. For the filtration $\{\overline{I^r J^s}\}_{r,s \in \mathbb{Z}}$, the coefficients of degree one of the normal Hilbert polynomial of I and J were studied by Rees in [14]. He showed that if R is an analytically unramified Cohen-Macaulay local ring of dimension 2, then $\overline{e}_{(1,0)} = \overline{e}_1(I)$ and $\overline{e}_{(0,1)} = \overline{e}_1(J)$ ([13, Theorem 1.2]). Here, $\overline{e}_{(1,0)}, \overline{e}_{(0,1)}$ are the first normal Bhattacharya coefficients and $\overline{e}_1(I)$ (resp. $\overline{e}_1(J)$) is the first normal Hilbert coefficient of I (resp. J). In [3] first author and A. Guerrieri proved that $e_{(d-1,0)} = e_1(I)$ in any Noetherian local ring of dimension d . This is not true in general (Example 4.10).

In this paper we express the difference $e_{(1,0)} - e_1(I)$ in terms of the length of modules which arise from the modified Koszul complex (Proposition 3.11). We show that $e_{(1,0)} \geq e_1(I)$ and $e_{(0,1)} \geq e_1(J)$ in a Cohen-Macaulay local ring of dimension 2 (Proposition 3.11). Moreover, equality holds true if and only if $\lambda_R(H_{(at_1, bt_2)}^2(\mathcal{R}')_{(0,0)}) < \infty$ for a joint reduction (a, b) of (I, J) satisfying superficial conditions (Theorem 4.6). By an example (Example 4.10) we show that for $d = 2$, $e_{(1,0)} - e_1(I)$ can be as large as possible.

We next address the problem of vanishing of $\lambda(H_{(at_1, bt_2)}^2(\mathcal{R}')_{(r,s)})$ ($r, s \geq 0$). For a bigraded filtration $\{\overline{I^r J^s}\}_{r,s \in \mathbb{Z}}$ the vanishing of $H_{(at_1, bt_2)}^2(\overline{\mathcal{R}'}_{(r,s)})$ has been studied in [9]. In this paper we give necessary and sufficient conditions for the vanishing of $H_{(at_1, bt_2)}^2(\mathcal{R}')_{(r,s)}$ in a Cohen-Macaulay local ring of dimension 2. We show that $H_{(at_1, bt_2)}^2(\mathcal{R}')_{(0,0)} = 0$ if and only if the joint reduction number of I^k and J^k is zero for $k \gg 0$ (Theorem 5.2). Replacing the filtration $\{I^r J^s\}_{r,s \in \mathbb{Z}}$ by the filtration $\{\overline{I^r J^s}\}_{r,s \in \mathbb{Z}}$ in Theorem 5.3 we recover the result of D. Rees which gives a relation between the normal Hilbert coefficients and joint reduction number [13, Theorem 2.5].

The paper is organised as follows. In section 2 we gather preliminary results needed in the subsequent sections. In section 3 we relate the Bhattacharya coefficients and the Hilbert coefficients. In section 4 we give equivalent conditions for $\lambda(H_{(at_1, bt_2)}^2(\mathcal{R}')_{(r,s)})$ to be finite for a Cohen-Macaulay local ring R of dimension 2 and a joint reduction (a, b) of (I, J) satisfying superficial conditions. In section 5 we give necessary and sufficient conditions for vanishing of $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)}$ for a Cohen-Macaulay local ring R of dimension 2.

Remark 1.4. For ideals $\mathcal{F}_{(r,s)}$ in R , we say that $\mathcal{F} = \{\mathcal{F}_{(r,s)}\}_{r,s \in \mathbb{Z}}$ is an (I, J) -filtration if for all integers r, s, m, n , $I^r J^s \subseteq \mathcal{F}_{(r,s)}$, $\mathcal{F}_{(r,s)} \mathcal{F}_{(m,n)} \subseteq \mathcal{F}_{(r+m, s+n)}$ and $\mathcal{F}_{(r,s)} \subseteq \mathcal{F}_{(m,n)}$ for $(r, s) \geq (m, n)$. We say \mathcal{F} is an *admissible* (I, J) -filtration if the extended Rees algebra $\mathcal{R}'(\mathcal{F}) = \bigoplus_{r,s \in \mathbb{Z}} \mathcal{F}_{(r,s)} t_1^r t_2^s$ is a finite module over \mathcal{R}' . All our proofs work for any admissible (I, J) -filtration \mathcal{F} . For the sake of convenience we work with the filtration $\mathcal{F} = \{I^r J^s\}_{r,s \in \mathbb{Z}}$.

We refer [10] for all undefined terms.

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2. PRELIMINARIES

Let (R, \mathfrak{m}) be a Noetherian local ring (R, \mathfrak{m}) of dimension d and I, J be \mathfrak{m} -primary ideals in R . For an indeterminate t over R , let $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n \subseteq R[t]$ be the Rees algebra of I . For fixed $r > 0$, consider the graded ring $\mathcal{R}(J) \otimes_R R/I^r = \bigoplus_{s \geq 0} J^s / J^s I^r$. Then for $s \gg 0$, $\lambda(\mathcal{R}(J) \otimes_R R/I^r)_s = \lambda(J^s / J^s I^r)$ is a polynomial of degree $d - 1$ in s . Similarly, for fixed $s > 0$ and $r \gg 0$, $\lambda(\mathcal{R}(I) \otimes_R R/J^s)_r = \lambda(I^r / I^r J^s)$ is a polynomial of degree $d - 1$ in r . Hence if $d = 2$, using (1.2), we have

$$\lambda\left(\frac{R}{I^r J^s}\right) = e(J) \binom{s+1}{2} - g_1(r)s + g_2(r) \quad \text{for fixed } r \geq 0 \text{ and } s \gg 0 \quad (2.1)$$

$$\lambda\left(\frac{R}{I^r J^s}\right) = e(I) \binom{r+1}{2} - h_1(s)r + h_2(s) \quad \text{for fixed } s \geq 0 \text{ and } r \gg 0 \quad (2.2)$$

for some integers $g_1(r), g_2(r), h_1(s), h_2(s)$. For $r \gg 0$ (resp. $s \gg 0$), $g_1(r), g_2(r)$ (resp. $h_1(s), h_2(s)$) are polynomials in r (resp. s). More precisely,

$$g_1(0) = e_1(J), \quad h_1(0) = e_1(I) \quad (2.3)$$

$$g_2(0) = e_2(J), \quad h_2(0) = e_2(I) \quad (2.4)$$

$$g_1(r) = -e_{(1,1)}r + e_{(0,1)} \text{ for } r \gg 0, \quad h_1(s) = -e_{(1,1)}s + e_{(1,0)} \text{ for } s \gg 0 \quad (2.5)$$

$$g_2(r) = e(I) \binom{r+1}{2} - e_{(1,0)}r + e_{(0,0)} \text{ for } r \gg 0 \quad (2.6)$$

$$h_2(s) = e(J) \binom{s+1}{2} - e_{(0,1)}s + e_{(0,0)} \text{ for } s \gg 0. \quad (2.7)$$

Let $r, s \in \mathbb{Z}, k \geq 1, a \in I$ and $b \in J$. Let $C_\bullet((a^k, b^k), r, s)$ denote the complex

$$C_\bullet((a^k, b^k), r, s) : 0 \longrightarrow \frac{R}{I^r J^s} \xrightarrow{\phi_1} \frac{R}{I^{r+k} J^s} \bigoplus \frac{R}{I^r J^{s+k}} \xrightarrow{\phi_0} \frac{R}{I^{r+k} J^{s+k}} \longrightarrow 0,$$

where the maps are induced by the Koszul complex $K_\bullet(a^k, b^k; R)$. Let $H_i((a^k, b^k), r, s)$ denote the i -th homology of the complex $C_\bullet((a^k, b^k), r, s)$. First author and A. Guerrieri derived a formula for $H_i((a^k, b^k), r, s)$, $i = 0, 1, 2$ [4].

Theorem 2.8. [4] *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . Let $a \in I$ and $b \in J$. Then for all $k \geq 1$ and $r, s \geq 0$,*

$$\begin{aligned} (a) \quad H_0((a^k, b^k), r, s) &= \frac{R}{I^{r+k} J^{s+k} + (a^k, b^k)} \\ (b) \quad H_2((a^k, b^k), r, s) &= \frac{(I^{r+k} J^s : (a^k)) \cap (I^r J^{s+k} : (b^k))}{I^r J^s} \\ (c) \quad \text{If } a, b \text{ is a regular sequence, then } H_1((a^k, b^k), r, s) &= \frac{(a^k, b^k) \cap I^{r+k} J^{s+k}}{a^k I^r J^{s+k} + b^k I^{r+k} J^s}. \end{aligned}$$

Let R be a local ring of dimension 2. For ideals I and J , we say (a, b) is a *joint reduction* of (I, J) if $a \in I$, $b \in J$ and

$$I^{r+1} J^{s+1} = a I^r J^{s+1} + b I^{r+1} J^s \text{ for some } r, s \text{ and hence for all } r, s \gg 0. \quad (2.9)$$

Let $a \in I$ and $b \in J$. We say (a, b) satisfies *superficial conditions* if the following equations hold true :

$$(a) \cap I^r J^s = a I^{r-1} J^s \text{ for } r \gg 0 \text{ and all } s \geq 0 \text{ and} \quad (2.10)$$

$$(b) \cap I^r J^s = b I^r J^{s-1} \text{ for } s \gg 0 \text{ and all } r \geq 0. \quad (2.11)$$

In [14], Rees proved that if the residue field of R is infinite, then there exist joint reductions satisfying superficial conditions.

3. BHATTACHARYA COEFFICIENTS AND HILBERT COEFFICIENTS

Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . In this section we relate first Bhattacharya coefficient $e_{(1,0)}$ (resp. $e_{(0,1)}$) to the first Hilbert coefficient $e_1(I)$ (resp. $e_1(J)$). We derive a formula for $e_{(1,0)}$ (resp. $e_{(0,1)}$) in terms of $e_1(I)$ (resp. $e_1(J)$) and modules that arise from the modified Koszul complex. In particular, we prove that $e_{(1,0)} \geq h_1(s) + se_{(1,1)}$ and $e_{(0,1)} \geq g_1(r) + re_{(1,1)}$ which gives $e_{(1,0)} \geq e_1(I)$ and $e_{(0,1)} \geq e_1(J)$. For fixed $r, s \geq 0$ and $k \gg 0$, we show that $\lambda\left(\frac{I^{r+k}J^{s+k}}{a^k I^r J^{s+k} + b^k I^{r+k} J^s}\right)$ is a polynomial in k of degree at most one, whose coefficients involve the difference of Bhattacharya and Hilbert coefficients. We show that $\lambda\left(\frac{I^k J^k}{a^k J^k + b^k I^k}\right)$ does not depend on the choice of the joint reduction chosen for $k \gg 0$.

First we study properties of $H_2((a^k, b^k), r, s)$. For this purpose we need the notion of Ratliff-Rush closure of (I, J) with respect to a joint reduction. The Ratliff-Rush closure of an ideal was introduced in [12]. In [7] the Ratliff-Rush closure for product of two ideals was computed using complete reductions.

Definition 3.1. Let I, J be \mathfrak{m} -primary ideals and let (a, b) be a joint reduction of (I, J) . For $r, s \geq 0$, we define the Ratliff-Rush closure of (I^r, J^s) with respect to (a, b)

$$rr_{(a,b)}(I^r, J^s) := \bigcup_{k \geq 1} (I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k).$$

Remark 3.2. For all $k \geq 1$, $(I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k) \subseteq (I^{r+k+1} J^s : a^{k+1}) \cap (I^r J^{s+k+1} : b^{k+1})$. As R is Noetherian, $rr_{(a,b)}(I^r, J^s) = (I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k)$ for some and hence for all $k \gg 0$.

Lemma 3.3. Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . Let (a, b) be a joint reduction of (I, J) . Then

- (a) For all $r, s \geq 0$ and $k \gg 0$, $H_2((a^k, b^k), r, s) = \frac{rr_{(a,b)}(I^r, J^s)}{I^r J^s}$ and hence is independent of k .
- (b) If in addition (a, b) satisfies superficial conditions, then for fixed $s \geq 0$ and $r \gg 0$ (resp. fixed $r \geq 0$ and $s \gg 0$), $H_2((a^k, b^k), r, s) = 0$ for all $k \geq 1$.

Proof. (a) For all $k \geq 1$, $(I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k) \subseteq (I^{r+k+1} J^s : a^{k+1}) \cap (I^r J^{s+k+1} : b^{k+1})$. As R is Noetherian $rr_{(a,b)}(I^r, J^s) = (I^{r+k} J^s : a^k) \cap (I^r J^{s+k} : b^k)$ for all $k \gg 0$. Hence, by Theorem 2.8, for $k \gg 0$,

$$H_2((a^k, b^k), r, s) = \frac{rr_{(a,b)}(I^r, J^s)}{I^r J^s}.$$

- (b) From (2.10), $(I^{r+k} J^s : a^k) = I^r J^s$ for all $k \geq 1$ and $r \gg 0$. Hence $H_2((a^k, b^k), r, s) = 0$.

□

Notation 3.4. For a joint reduction (a, b) of (I, J) , we set $L_{(a,b)}(r, s; k) := \frac{I^{r+k} J^{s+k}}{a^k I^r J^{s+k} + b^k I^{r+k} J^s}$.

Lemma 3.5. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . Let (a, b) be a joint reduction of (I, J) . Then for all $r, s \geq 0$ and $k \geq 1$,*

$$\begin{aligned} & \lambda(H_0((a^k, b^k), r, s)) - \lambda(H_1((a^k, b^k), r, s)) + \lambda(H_2((a^k, b^k), r, s)) \\ &= k^2 e_{(1,1)} - \lambda(L_{(a,b)}(r, s; k)) + \lambda(H_2((a^k, b^k), r, s)). \end{aligned}$$

Proof. Using Theorem 2.8, for all $k \geq 1$, we have

$$\begin{aligned} & \lambda(H_0((a^k, b^k), r, s)) - \lambda(H_1((a^k, b^k), r, s)) + \lambda(H_2((a^k, b^k), r, s)) \\ &= \lambda\left(\frac{R}{(a^k, b^k) + I^{r+k}J^{s+k}}\right) - \lambda\left(\frac{(a^k, b^k) \cap I^{r+k}J^{s+k}}{a^k I^r J^{s+k} + b^k I^{r+k} J^s}\right) + \lambda(H_2((a^k, b^k), r, s)) \\ &= \lambda\left(\frac{R}{(a^k, b^k)}\right) - \lambda\left(\frac{(a^k, b^k) + I^{r+k}J^{s+k}}{(a^k, b^k)}\right) - \lambda\left(\frac{(a^k, b^k) \cap I^{r+k}J^{s+k}}{a^k I^r J^{s+k} + b^k I^{r+k} J^s}\right) + \lambda(H_2((a^k, b^k), r, s)) \\ &= \lambda\left(\frac{R}{(a^k, b^k)}\right) - \lambda(L_{(a,b)}(r, s; k)) + \lambda(H_2((a^k, b^k), r, s)) \\ &= k^2 e_{(1,1)} - \lambda(L_{(a,b)}(r, s; k)) + \lambda(H_2((a^k, b^k), r, s)) \quad (\text{from [14, Theorem 2.4]}). \end{aligned}$$

□

Lemma 3.6. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . Let (a, b) be a joint reduction of (I, J) satisfying superficial conditions.*

(a) *Fix $r \geq 0$. Then for all $s \gg 0$,*

$$\lambda(L_{(a,b)}(r, s; 1)) = g_1(r+1) - g_1(r) + e_{(1,1)}.$$

In particular, for fixed $r \geq 0$ and $s \gg 0$, $\lambda(L_{(a,b)}(r, s; 1))$ is independent of s .

(b) *Fix $s \geq 0$. Then for all $r \gg 0$,*

$$\lambda(L_{(a,b)}(r, s; 1)) = h_1(s+1) - h_1(s) + e_{(1,1)}.$$

In particular, for fixed $s \geq 0$ and $r \gg 0$, $\lambda(L_{(a,b)}(r, s; 1))$ is independent of r .

Proof. It is enough to prove (a) as the proof of (b) is similar.

For fixed $r \geq 0$ and $s \gg 0$, we have

$$\begin{aligned} -g_1(r+1) + g_1(r) &= \lambda\left(\frac{R}{I^{r+1}J^{s+1}}\right) - \left[\lambda\left(\frac{R}{I^{r+1}J^s}\right) + \lambda\left(\frac{R}{I^r J^{s+1}}\right)\right] + \lambda\left(\frac{R}{I^r J^s}\right) \quad (\text{from (2.1)}), \\ &= \lambda(H_0((a, b), r, s)) - \lambda(H_1((a, b), r, s)) + \lambda(H_2((a, b), r, s)) \\ &= e_{(1,1)} - \lambda(L_{(a,b)}(r, s; 1)) \quad (\text{from Lemma 3.3(b) and Lemma 3.5}). \end{aligned}$$

Therefore $\lambda(L_{(a,b)}(r, s; 1)) = e_{(1,1)} + g_1(r+1) - g_1(r)$ for $s \gg 0$. As the right hand side is independent of s , $\lambda(L_{(a,b)}(r, s; 1))$ is independent of s for $s \gg 0$. □

Notation 3.7. For $i, j \geq 0$, we set

$$\alpha(i) := g_1(i+1) - g_1(i) + e_{(1,1)}. \quad (3.8)$$

$$\beta(j) := h_1(j+1) - h_1(j) + e_{(1,1)}. \quad (3.9)$$

Remark 3.10. (a) With the assumptions as in Lemma 3.6, we get that for $s \gg 0$, $\lambda(L_{(a,b)}(i, s; 1)) = \alpha(i)$ and for $r \gg 0$, $\lambda(L_{(a,b)}(r, j; 1)) = \beta(j)$. Hence $\alpha(i)$ and $\beta(j)$ are non-negative.

(b) From (2.5) it follows that for $i \gg 0$, $\alpha(i) = 0$. Similarly, for $j \gg 0$, $\beta(j) = 0$.

Proposition 3.11. Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2. Let I and J be \mathfrak{m} -primary ideals in R . Then

(a) $e_{(0,1)} = e_1(J) + \sum_{i \geq 0} \alpha(i)$. In particular, $e_{(0,1)} \geq e_1(J)$.

(b) $e_{(1,0)} = e_1(I) + \sum_{j \geq 0} \beta(j)$. In particular, $e_{(1,0)} \geq e_1(I)$.

Proof. (a) From (3.8) we get

$$\sum_{i=0}^{r-1} [-g_1(i+1) + g_1(i)] = re_{(1,1)} - \sum_{i=0}^{r-1} \alpha(i).$$

Hence for all $r \geq 0$,

$$-g_1(r) + g_1(0) = re_{(1,1)} - \sum_{i=0}^{r-1} \alpha(i). \quad (3.12)$$

Since, for $r \gg 0$, $g_1(r) = e_{(0,1)} - re_{(1,1)}$, $g_1(0) = e_1(J)$ and $\alpha(r) = 0$, substituting in (3.12) we get

$$e_{(0,1)} = e_1(J) + \sum_{i \geq 0} \alpha(i).$$

This proves (a).

(b) Similarly, replacing $g_1(r)$ by $h_1(s)$, $e_{(0,1)}$ by $e_{(1,0)}$ and $\alpha(i)$ by $\beta(j)$ in the proof of (a) we get (b).

□

Proposition 3.13. Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . Then

(a) For all $r \geq 0$, $e_{(0,1)} - g_1(r) - re_{(1,1)} \geq 0$.

(b) For all $s \geq 0$, $e_{(1,0)} - h_1(s) - se_{(1,1)} \geq 0$.

Proof. From (3.12),

$$\begin{aligned}
e_{(0,1)} - g_1(r) - re_{(1,1)} &= e_{(0,1)} - g_1(0) - \sum_{i=0}^{r-1} \alpha(i) \\
&= e_{(0,1)} - e_1(J) - \sum_{i=0}^{r-1} \alpha(i) \quad (\text{as } g_1(0) = e_1(J)) \\
&\geq e_{(0,1)} - e_1(J) - \sum_{i \geq 0} \alpha(i) \\
&= 0 \quad (\text{by Proposition 3.11}).
\end{aligned} \tag{3.14}$$

□

In the next proposition we give an explicit formula for $\lambda(L_{(a,b)}(r, s; k))$ in terms of the Bhattacharya and Hilbert coefficients.

Proposition 3.15. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . Let (a, b) be a joint reduction of (I, J) . Fix $r, s \geq 0$. Then for $k \gg 0$,*

$$\begin{aligned}
&\lambda(L_{(a,b)}(r, s; k)) \\
&= (e_{(0,1)} - g_1(r) - e_{(1,1)}r)(s + k) + (e_{(1,0)} - h_1(s) - e_{(1,1)}s)(r + k) \\
&\quad - e_2(IJ) + g_2(r) + h_2(s) - \lambda(R/I^r J^s) + rse_{(1,1)} + \lambda\left(\frac{rr_{(a,b)}(I^r, J^s)}{I^r J^s}\right).
\end{aligned}$$

In particular, for $k \gg 0$, $\lambda(L_{(a,b)}(r, s; k))$ is a polynomial in k of degree at most 1.

Proof. From Lemma 3.5 for $k \gg 0$, we have

$$\begin{aligned}
&\lambda(L_{(a,b)}(r, s; k)) \\
&= k^2 e_{(1,1)} + \lambda(H_2((a^k, b^k), r, s)) - \sum_{i=0}^2 (-1)^i \lambda(H_i((a^k, b^k), r, s)) \\
&= k^2 e_{(1,1)} + \lambda(H_2((a^k, b^k), r, s)) - \left[\lambda\left(\frac{R}{I^{r+k} J^{s+k}}\right) - \lambda\left(\frac{R}{I^{r+k} J^s}\right) - \lambda\left(\frac{R}{I^r J^{s+k}}\right) + \lambda\left(\frac{R}{I^r J^s}\right) \right] \\
&= k^2 e_{(1,1)} + \lambda(H_2((a^k, b^k), r, s)) \\
&\quad - \left[e_{(1,1)}(r + k)(s + k) - e_{(1,0)}(r + k) - e_{(0,1)}(s + k) + e_{(0,0)} + h_1(s)(r + k) - h_2(s) \right. \\
&\quad \left. + g_1(r)(s + k) - g_2(r) + \lambda\left(\frac{R}{I^r J^s}\right) \right] \quad [\text{from (1.2), (2.1), (2.2)}] \\
&= (e_{(0,1)} - g_1(r) - e_{(1,1)}r)(s + k) + (e_{(1,0)} - h_1(s) - e_{(1,1)}s)(r + k) \\
&\quad - e_2(IJ) + g_2(r) + h_2(s) - \left(\frac{R}{I^r J^s}\right) + rse_{(1,1)} + \lambda\left(\frac{rr_{(a,b)}(I^r, J^s)}{I^r J^s}\right) [\text{by Lemma 3.3 (a)}].
\end{aligned}$$

□

As a corollary we give a relation between $L_{(a,b)}(0,0;k)$, the Bhattacharya coefficients and the Hilbert coefficients.

Proposition 3.16. *With the assumptions as in Proposition 3.15, for $k \gg 0$,*

$$\lambda(L_{(a,b)}(0,0;k)) = (e_{(0,1)} - e_1(J))k + (e_{(1,0)} - e_1(I))k - e_2(IJ) + e_2(J) + e_2(I).$$

In particular, $\lambda(L_{(a,b)}(0,0;k))$ does not depend on the choice of the joint-reduction chosen for $k \gg 0$.

Proof. For $r = s = 0$, $\lambda\left(\frac{rr_{(a,b)}(I^r, J^s)}{I^r J^s}\right) = 0$. As $g_1(0) = e_1(J)$ and $h_1(0) = e_1(I)$, substituting in Proposition 3.15 we get the result. \square

4. LOCAL COHOMOLOGY OF BIGRADED REES ALGEBRAS

Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . In Section 3 we showed that $e_{(1,0)} \geq e_1(I)$ and $e_{(0,1)} \geq e_1(J)$. In this section we give an equivalent criterion for the equality to hold true in terms of local cohomology modules of the bigraded extended Rees algebras. We show that if $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)}) < \infty$ for some $r, s \geq 0$, then $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(p,q)}) < \infty$ for all $p \geq r$ and $q \geq s$, where (a, b) is a joint reduction of (I, J) satisfying superficial conditions. We give an example to show that $e_{(0,1)} \neq e_1(J)$ in general and that the difference $e_{(0,1)} - e_1(J)$ can be as large as possible.

For $a \in I, b \in J$, consider the Koszul co-complex

$$K^\bullet((at_1)^k, (bt_2)^k; \mathcal{R}') : 0 \longrightarrow \mathcal{R}' \xrightarrow{\alpha_k} \mathcal{R}'(k, 0) \oplus \mathcal{R}'(0, k) \xrightarrow{\beta_k} \mathcal{R}'(k, k) \longrightarrow 0,$$

where the maps are defined as,

$$\alpha_k(1) = ((at_1)^k, (bt_2)^k) \text{ and } \beta_k(u, v) = -(bt_2)^k u + (at_1)^k v.$$

Then for all i

$$H_{(at_1, bt_2)}^i(\mathcal{R}') = \varinjlim_k H^i(K^\bullet((at_1)^k, (bt_2)^k; \mathcal{R}')) \quad [2, \text{Theorem 5.2.9}]. \quad (4.1)$$

In [9] the second author and Verma derived a formula for $[H_{(at_1, bt_2)}^2(\overline{\mathcal{R}'})]_{(r,s)}$ in terms of normal Hilbert coefficients, where (a, b) is a good joint reduction of the filtration $\{\overline{I^r J^s}\}_{r,s \in \mathbb{Z}}$. In following theorem we recover some results for $H_{(at_1, bt_2)}^2(\mathcal{R}')$, where (a, b) is a joint reduction of the filtration (I, J) satisfying superficial conditions.

Theorem 4.2. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2. Let I and J be \mathfrak{m} -primary ideals in R and (a, b) be a joint reduction of (I, J) . Then for all $r, s \geq 0$,*

(a)

$$[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)} \cong \varinjlim_k L_{(a,b)}(r, s; k).$$

(b) If in addition (a, b) satisfies superficial conditions, then for $k \gg 0$, the maps

$$\mu_k : L_{(a,b)}(r, s; k) \xrightarrow{\cdot(ab)} L_{(a,b)}(r, s; k+1)$$

are injective.

Proof. (a) The proof follows from (4.1).

(b) Let $x \in I^{r+k}J^{s+k}$ be such that $\mu_k(\bar{x}) = 0$. Then $xab = a^{k+1}p + b^{k+1}q$ for some $p \in I^rJ^{s+k+1}$ and $q \in I^{r+k+1}J^s$. Hence for $k \gg 0$, $q \in (a) \cap I^{r+k+1}J^s = aI^{r+k}J^s$. Therefore $q = aq'$ for some $q' \in I^{r+k}J^s$. Similarly, for $k \gg 0$, $p = bp'$ for some $p' \in I^rJ^{s+k}$. Hence

$$x = a^k p' + b^k q' \in a^k I^r J^{s+k} + b^k I^{r+k} J^s.$$

Thus $\bar{x} = 0$ and hence μ_k is injective for all $k \gg 0$. □

The map μ_k defined in Theorem 4.2 need not be surjective for $k \gg 0$. If μ_k is not surjective, then from Proposition 3.15 and Theorem 4.2, $\lambda_R([H_{(at_1, bt_2)}^2(\mathcal{R}')](r, s))$ is infinite for a joint reduction (a, b) of (I, J) satisfying superficial conditions. In Theorem 4.3 we give equivalent conditions for $\lambda_R([H_{(at_1, bt_2)}^2(\mathcal{R}')](r, s))$ to be finite.

Theorem 4.3. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . Let $r, s \geq 0$. Then following statements are equivalent.*

(a) *There exists a joint reduction (a, b) of (I, J) satisfying superficial conditions such that*

$$\lambda_R([H_{(at_1, bt_2)}^2(\mathcal{R}')](r, s)) < \infty,$$

(b) $e_{(0,1)} = g_1(r) + re_{(1,1)}$ and $e_{(1,0)} = h_1(s) + se_{(1,1)}$,

(c) *There exists a joint reduction (a, b) of (I, J) satisfying superficial conditions such that*

$$\begin{aligned} I^i J^m &= aI^{i-1}J^m + bI^i J^{m-1} && \text{for } i > r \text{ and } m \gg 0 \text{ and} \\ I^m J^i &= aI^{m-1}J^i + bI^m J^{i-1} && \text{for } m \gg 0 \text{ and } i > s. \end{aligned}$$

If any of the above equivalent conditions hold true, then for any joint reduction (a, b) of (I, J) satisfying superficial conditions,

$$\lambda_R([H_{(at_1, bt_2)}^2(\mathcal{R}')](r, s)) = -e_2(IJ) + g_2(r) + h_2(s) - \lambda(R/I^r J^s) + rse_{(1,1)} + \lambda\left(\frac{rr_{(a,b)}(I^r, J^s)}{I^r J^s}\right).$$

Proof. (a) \Rightarrow (b) : By Proposition 3.15, for $k \gg 0$, $\lambda(L_{(a,b)}(r, s; k))$ is a polynomial in k of degree at most 1. By Theorem 4.2, for all $k \gg 0$,

$$\lambda(L_{(a,b)}(r, s; k)) \leq \lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')](r, s)) < \infty.$$

Hence $\lambda(L_{(a,b)}(r, s; k))$ is a constant for $k \gg 0$. This implies that $(e_{(0,1)} - g_1(r) - re_{(1,1)}) + (e_{(1,0)} - h_1(s) - se_{(1,1)}) = 0$. Since $e_{(0,1)} - g_1(r) - re_{(1,1)}$ and $e_{(1,0)} - h_1(s) - se_{(1,1)}$ are non-negative (by Proposition 3.13),

$$e_{(0,1)} - g_1(r) - re_{(1,1)} = e_{(1,0)} - h_1(s) - se_{(1,1)} = 0.$$

(b) \Rightarrow (c) : By our assumption we have

$$\begin{aligned} 0 &= e_{(0,1)} - g_1(r) - re_{(1,1)} \\ &= e_{(0,1)} - e_1(J) - \sum_{i=0}^{r-1} \alpha(i) \quad (\text{from (3.12)}) \\ &\geq e_{(0,1)} - e_1(J) - \sum_{i \geq 0} \alpha(i) \\ &= 0 \quad (\text{from Proposition 3.11}). \end{aligned} \tag{4.4}$$

Hence the inequality in (4.4) is an equality and we get $\alpha(i) = 0$ for $i \geq r$.

Let (a, b) be a joint reduction of (I, J) satisfying superficial conditions. Then from Remark 3.10(a),

$$\lambda(L_{(a,b)}(i, m; 1)) = 0 \text{ for all } i \geq r \text{ and for all } m \gg 0.$$

Hence for $i > r$ and $m \gg 0$, $I^i J^m = aI^{i-1}J^m + bI^i J^{m-1}$.

Similarly, $e_{(1,0)} = h_1(s) + se_{(1,1)}$ implies $I^m J^i = aI^{m-1}J^i + bI^m J^{i-1}$ for $m \gg 0$ and $i > s$.

(c) \Rightarrow (a) : By our assumption, $L_{(a,b)}(i, m; 1) = 0$ for $i \geq r$ and $m \gg 0$. Therefore, by Remark 3.10(a), $\alpha(i) = 0$ for $i \geq r$. Hence

$$\begin{aligned} e_{(0,1)} &= e_1(J) + \sum_{i=0}^{r-1} \alpha(i) \quad (\text{Proposition 3.11}) \\ &= g_1(r) + re_{(1,1)} \quad (\text{from (3.12)}) \end{aligned}$$

Similarly, $e_{(1,0)} = h_1(s) + se_{(1,1)}$. Substituting for $e_{(0,1)}$ and $e_{(1,0)}$ in Proposition 3.15, we get $\lambda(L_{(a,b)}(r, s; k))$ is a constant for $k \gg 0$. Since μ_k is injective for $k \gg 0$, we conclude that μ_k is an isomorphism for $k \gg 0$. Hence for $k \gg 0$,

$$[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)} \cong L_{(a,b)}(r, s; k) < \infty.$$

□

Remark 4.5. The proof of Theorem 4.3 shows that if condition (a) (resp. (c)) is satisfied for a joint reduction (a, b) of (I, J) satisfying superficial conditions then condition (a) (resp. (c)) is satisfied for every joint reduction (a, b) of (I, J) satisfying superficial conditions.

As a corollary we give equivalent conditions for $\lambda_R([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)}) < \infty$.

Theorem 4.6. *With the assumptions as in Theorem 4.3, the following statements are equivalent.*

(a) There exists a joint reduction (a, b) of (I, J) satisfying superficial conditions such that

$$\lambda_R([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)}) < \infty,$$

(b) $e_{(1,0)} = e_1(I)$ and $e_{(0,1)} = e_1(J)$,

(c) There exists a joint reduction (a, b) of (I, J) satisfying superficial conditions such that

$$\begin{aligned} I^i J^m &= aI^{i-1} J^m + bI^i J^{m-1} && \text{for } i > 0 \text{ and } m \gg 0 \text{ and} \\ I^m J^i &= aI^{m-1} J^i + bI^m J^{i-1} && \text{for } m \gg 0 \text{ and } i > 0. \end{aligned}$$

If any of the above equivalent conditions hold true, then for any joint reduction (a, b) of (I, J) satisfying superficial conditions,

$$\lambda_R([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)}) = -e_2(IJ) + e_2(I) + e_2(J).$$

Proof. Put $r = s = 0$ in Theorem 4.3. □

Theorem 4.7. *With the assumptions as in Theorem 4.3, for a joint reduction (a, b) of (I, J) satisfying superficial conditions if $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)}) < \infty$ for some $r, s \geq 0$, then $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(p,q)}) < \infty$ for all $p \geq r$ and $q \geq s$.*

Proof. Suppose $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)}) < \infty$ for some $r, s \geq 0$. Then by Remark 4.5, statement (c) of Theorem 4.3 holds true for all $p \geq r$ and $q \geq s$. By using Theorem 4.3 once again we get $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(p,q)}) < \infty$ for all $p \geq r$ and $q \geq s$. □

In what follows we give an example to show that $e_{(1,0)} \neq e_1(I)$ and hence $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)})$ is not finite.

An ideal $J \subseteq I$ is called a *reduction* of I if $J I^n = I^{n+1}$ for some n . We say J is a *minimal reduction* of I if whenever $K \subseteq J$ and K is a reduction of I , then $K = J$ [11]. The *reduction number* of I with respect to a minimal reduction J of I is defined as

$$r_J(I) := \min\{n \geq 0 \mid J I^n = I^{n+1}\}.$$

The *reduction number* of I denoted by $r(I)$ is defined to be the minimum of $r_J(I)$ where J varies over all minimal reductions of I .

Proposition 4.8. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I be an \mathfrak{m} -primary ideal in R with $r(I) \geq 1$. Let $J = (x, y)$ be a minimal reduction of I such that (x, y) is a superficial sequence for I and $I = (x, y, a_1, \dots, a_{\mu-2})$, where μ denotes the minimal number of generators of I . Then*

- (a) $I J^n \neq y J^n + x I J^{n-1}$ for all $n \geq 1$.
- (b) $e_{(0,1)} \neq e_1(J)$.

Proof. (a) Note that $\mu > 2$ by [8, Theorem 3.21]. To prove the lemma it is enough to show that

$$y^n a_i \notin yJ^n + xIJ^{n-1} \text{ for all } i = 1, \dots, \mu - 2.$$

Suppose $y^n a_i \in yJ^n + xIJ^{n-1}$ for some i . Inductively, for all $n \geq 1$, we have

$$\begin{aligned} IJ^n &= (x, y)^{n+1} + (x, y)^n(a_1, \dots, a_{\mu-2}) \text{ and} \\ yJ^n + xIJ^{n-1} &= (x, y)^{n+1} + x(x, y)^{n-1}(a_1, \dots, a_{\mu-2}). \end{aligned} \quad (4.9)$$

Hence from (4.9),

$$y^n a_i = \sum_{k=0}^{n+1} x^k y^{n+1-k} r_k + x \sum_{k=0}^{n-1} \left(\sum_{j=1}^{\mu-2} s_{kj} x^k y^{n-1-k} a_j \right)$$

where $r_k, s_{kj} \in R$. This implies that

$$y^n(a_i - xr_1 - yr_0) \in (x).$$

As x, y is a regular sequence in R ,

$$a_i - xr_1 - yr_0 \in (x).$$

Therefore $a_i \in (x, y)$ which contradicts our assumption that I is minimally generated by $(x, y, a_1, \dots, a_{\mu-2})$.

(b) Suppose $e_{(0,1)} = e_1(J)$. Then by Proposition 3.11, $\alpha(0) = 0$. By Remark 3.10 we get

$$IJ^n = yJ^n + xIJ^{n-1} \text{ for } n \gg 0,$$

which contradicts (a). □

We give an example to show that the difference $e_{(0,1)} - e_1(J)$ can be as large as possible.

Example 4.10. Let $R = k[[x, y]]$, $\mathfrak{m} = (x, y)$, $I = \mathfrak{m}^l$, $J = (x^l, y^l)$, $l \geq 2$. Then by Proposition 4.8, $e_{(0,1)} \neq e_1(J)$. We explicitly calculate $e_{(0,1)} - e_1(J)$. For all $r, s \geq 1$,

$$\begin{aligned} \lambda\left(\frac{R}{I^r J^s}\right) &= \lambda\left(\frac{R}{\mathfrak{m}^{l(r+s)}}\right) \\ &= \binom{l(r+s)+1}{2} \\ &= l^2 \binom{r+1}{2} + l^2 rs + l^2 \binom{s+1}{2} - \binom{l}{2} r - \binom{l}{2} s. \end{aligned}$$

As J is a parameter ideal $e_1(J) = 0$. Hence $e_{(0,1)} - e_1(J) = \binom{l}{2}$. Note that (x^l, y^l) is a joint reduction of (I, J) satisfying superficial conditions. Therefore by Theorem 4.6 and Remark 4.5, $\lambda_R([H_{(x^l t_1, y^l t_2)}^2(\mathcal{R}')]_{(0,0)})$ is not finite. □

We give examples for which $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)}) < \infty$.

Definition 4.11. [16] We say that I and J have *joint reduction number zero*, denoted as $r(I|J) = 0$, if there exists a joint reduction (a, b) of (I, J) such that $IJ = aJ + bI$.

Remark 4.12. From [15, Theorem 3.2] it follows that if $r(I|J) = 0$ for \mathfrak{m} -primary ideals I and J in a Cohen-Macaulay local ring of dimension 2 with infinite residue field then the condition $IJ = aJ + bI$ holds for every joint reduction of I and J .

Example 4.13. (a) Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R such that $r(I|J) = 0$. Then for any joint reduction (a, b) of (I, J) , $L_{(a,b)}(r, s; k) = 0$ for all $r, s \geq 0$ and $k \geq 1$. Hence $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)} = 0$ for all $r, s \geq 0$ and any joint reduction (a, b) of (I, J) .

(b) Let (R, \mathfrak{m}) be a regular local ring of dimension two and let I, J be complete ideals (i.e. $\bar{I} = I$ and $\bar{J} = J$) in R . Then $IJ = aJ + bI$ for any joint reduction (a, b) of (I, J) [15, Theorem 2.1]. Hence $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)} = 0$ for all $r, s \geq 0$.

(c) Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I be an \mathfrak{m} -primary ideal of R . Let $J = I$. Then $e_{(1,0)} = e_{(0,1)} = e_1(I)$. Hence for any reduction (a, b) of I satisfying superficial conditions, $\lambda([H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)}) = e_2(I) < \infty$. (Theorem 4.6).

5. VANISHING OF LOCAL COHOMOLOGY MODULES

Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . In this section we give necessary and sufficient conditions for vanishing of $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r,s)}$ in terms of Bhattacharya and Hilbert coefficients. For any ideal I , let $G(I) := \bigoplus_{n \geq 0} I^n / I^{n+1}$ be the associated graded ring of I . We show that if $\text{depth } G(I)$ and $\text{depth } G(J) \geq 1$, then there exists a joint reduction (a, b) of (I, J) such that $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)} = 0$ if and only if $r(I|J) = 0$. We give an example to show that the result need not be true if either $\text{depth } G(I) = 0$ or $\text{depth } G(J) = 0$.

Theorem 5.1. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . For fixed $r_0, s_0 \geq 0$, the following statements are equivalent.*

- (a) *For every joint reduction (a, b) of (I, J) , $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r_0, s_0)} = 0$,*
- (b) *$e_{(0,1)} = g_1(r_0) + r_0 e_{(1,1)}$, $e_{(1,0)} = h_1(s_0) + s_0 e_{(1,1)}$ and $e_2(IJ) = g_2(r_0) + h_2(s_0) - \lambda(R/I^{r_0}J^{s_0}) + r_0 s_0 e_{(1,1)} + \lambda\left(\frac{rr_{(a,b)}(I^{r_0}, J^{s_0})}{I^{r_0}J^{s_0}}\right)$,*
- (c) *For every joint reduction (a, b) of (I, J)*

$$I^{r_0+k}J^{s_0+k} = a^k I^{r_0}J^{s_0+k} + b^k I^{r_0+k}J^{s_0} \text{ for } k \gg 0.$$

Proof. (a) \Rightarrow (b) : Put $r = r_0$ and $s = s_0$ in Theorem 4.3.

(b) \Rightarrow (c) : By Proposition 3.15, for every joint reduction (a, b) of (I, J) , $L_{(a,b)}(r_0, s_0; k) = 0$ for $k \gg 0$. Hence

$$I^{r_0+k}J^{s_0+k} = a^k I^{r_0}J^{s_0+k} + b^k I^{r_0+k}J^{s_0} \text{ for } k \gg 0.$$

(c) \Rightarrow (a) : Let (a, b) be a joint reduction of (I, J) . Then by assumption, $L_{(a,b)}(r_0, s_0; k) = 0$ for $k \gg 0$. By Theorem 4.2,

$$[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(r_0, s_0)} \cong \varinjlim_k L_{(a,b)}(r_0, s; k) = 0.$$

□

As a corollary we show that there exists a joint reduction (a, b) of (I, J) such that $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)} = 0$ if and only if $r(I^k | J^k) = 0$ for $k \gg 0$.

Theorem 5.2. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . The following statements are equivalent.*

- (a) *For every joint reduction (a, b) of (I, J) , $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)} = 0$,*
- (b) *$e_{(1,0)} = e_1(I)$, $e_{(0,1)} = e_1(J)$ and $e_2(IJ) = e_2(I) + e_2(J)$,*
- (c) *I^k and J^k have joint reduction number zero for $k \gg 0$.*

Proof. Put $r_0 = s_0 = 0$ in Theorem 5.1. □

Theorem 5.3. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension 2 and I, J be \mathfrak{m} -primary ideals in R . Assume that $\text{depth } G(I), \text{depth } G(J) \geq 1$. Then following statements are equivalent.*

- (a) *For every joint reduction (a, b) of (I, J) , $[H_{(at_1, bt_2)}^2(\mathcal{R}')]_{(0,0)} = 0$,*
- (b) *$e_{(1,0)} = e_1(I)$, $e_{(0,1)} = e_1(J)$ and $e_2(IJ) = e_2(I) + e_2(J)$,*
- (c) *I and J have joint reduction number zero.*

Proof. (a) \Rightarrow (b): Follows from Theorem 5.2.

(b) \Rightarrow (c) : By [14, Lemma 1.2] there exists a joint reduction (a, b) of (I, J) , $a \in I \setminus \mathfrak{m}I$ and $b \in J \setminus \mathfrak{m}J$, satisfying superficial conditions. Let a^* (resp. b^*) denotes the image of a (resp. b) in $[G(I)]_1$ (resp. $[G(J)]_1$). Since $\text{depth } G(I)$ (resp. $\text{depth } G(J)$) ≥ 1 , a^* (resp. b^*) is a nonzerodivisor in $G(I)$ (resp. $G(J)$), by [6, Lemma 2.1]. Hence

$$(a) \cap I^n = aI^{n-1} \text{ and } (b) \cap J^n = bJ^{n-1} \text{ for all } n > 0.$$

By Theorem 5.2, $I^k J^k = a^k J^k + b^k I^k$ for $k \gg 0$, say, for $k \geq N$. We prove that $I^k J^k = a^k J^k + b^k I^k$ for all $k \geq 1$. First we show that $I^{N-1} J^k = a^{N-1} J^k + b^k I^{N-1}$ for all $k \geq 1$. Let $x \in I^{N-1} J^k$. Then $ax \in I^N J^k$. Let $ax = a^N p + b^k q$ for some $p \in J^k$ and $q \in I^N$. Then $q \in (a) \cap I^N = aI^{N-1}$. Let $q = aq'$ for some $q' \in I^{N-1}$. Thus

$$x = a^{N-1} p + b^k q' \in a^{N-1} J^k + b^k I^{N-1}.$$

Similar argument shows that $I^k J^{N-1} = a^k J^{N-1} + b^{N-1} I^k$ for all $k \geq 1$. Continuing as above we get that $I^k J^k = a^k J^k + b^k I^k$ for all $k \geq 1$. Hence I and J have joint reduction number zero.

(c) \Rightarrow (a): Since $r(I|J) = 0$, by induction on k , we get $r(I^k|J^k) = 0$. Hence the result follow from Theorem 5.2. \square

Remark 5.4. Let $\overline{G}(I) = \bigoplus_{n \geq 1} \overline{I^n}/\overline{I^{n+1}}$. Then $\text{depth } \overline{G}(I), \overline{G}(J) \geq 1$ in an analytically unramified local ring. Hence replacing the filtration $\{I^r J^s\}_{r,s \in \mathbb{Z}}$ by $\{\overline{I^r J^s}\}_{r,s \in \mathbb{Z}}$ in Theorem 5.3, we recover Rees' theorem [13, Theorem 2.5]. (See Remark 1.4 and Theorem [13, Theorem 1.2]).

We give an example to show that the Theorem 5.3 need not be true if $\text{depth } G(I) = 0$ or $\text{depth } G(J) = 0$.

Example 5.5. Let $R = k[[x, y]]$, $I = (x^4, x^3y, xy^3, y^4)$ and $\mathfrak{m} = (x, y)$. Then $K = (x^4, y^4)$ is a minimal reduction of I and $x^2y^2 \in I^2 : K \setminus I$. Hence $\text{depth } G(I) = 0$ ([5, Corollary 3.3]). Moreover, (x^4, y) is a joint reduction of (I, \mathfrak{m}) and $I^2 \mathfrak{m}^2 = x^8 \mathfrak{m}^2 + y^2 I^2$. Hence $r(I^2|\mathfrak{m}^2) = 0$. By Theorem 5.2, $[H^2_{(at_1, bt_2)}(\mathcal{R}')]_{(0,0)} = 0$. As $x^2y^3 \in \mathfrak{m}I \setminus x^4 \mathfrak{m} + yI$, $r(I|\mathfrak{m}) \neq 0$. (See Remark 4.12.)

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